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Fast transient thermal analysis of Fourier and non-Fourier heat conduction

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Abstract

In this paper, asymptotic waveform evaluation (AWE) has been successfully used for fast transient characterization of Fourier and non-Fourier heat conduction. The Fourier and non-Fourier equations are reduced to a system of linear differential equations, respectively, using finite element method and then solved with AWE. Besides providing equivalent accuracy in its solution, it is also shown that AWE is at least three orders faster in term of computational time as compared to conventional iterative solvers. Its accuracy is also independent of the time step used and it has the capability of providing local transient solution. However, the moment matching process in AWE is inherently ill-conditioned and thus may yield unstable response even for stable system. This numerical instability is addressed and two stability schemes are also successfully implemented to yield stable and accurate solutions from AWE. The limitation of AWE is also discussed.
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Keywords: Asymptotic waveform evaluation (AWE); Fourier; Non-Fourier; Transient; Conduction

1. Introduction

Asymptotic waveform evaluation (AWE), which has been used for fast transient circuit simulation, is based on the concept of approximating the original system with a reduced order system. The inspiration of AWE came from Rubinstein et al. [1], where RC-tree networks were estimated using efficient Elmore delay approach. However, these estimates were not always accurate. A second breakthrough came from the work of McCormick [2], in which he has used the interconnect circuit moments to form a lower order circuit models to predict transient responses accurately. The efforts of these authors lead to the formalization and generalization of AWE algorithms [3,4].

For more than a decade, extensive works on AWE has been carried out. AWE has been successfully applied for fast transient circuit simulation [5–7]. AWE also has a lot of successes in electromagnetic simulations. However, there are only two papers available on the application of AWE in transient thermal simulation. Da et al. [8] have published the first paper on thermal analysis of PCB using AWE scheme, but the details of incorporating the initial conditions were not addressed. Then, Ooi et al. [9] has successfully extended the AWE algorithm to incorporate the initial conditions. They created a generalized formulation using the concept of zero state response and zero input response, which is used in control system. However, Ooi et al. [9] did not address the inherent numerical instability of AWE, which may yield incorrect solutions. Both papers

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Nomenclature

k	residue	Z_q	dimensionless phase lag for heat flux
M	moment	β	dimensionless time
p	pole	δ	dimensionless distance x
t	time (s)	ε	dimensionless distance y
T	temperature (°C)	θ	dimensionless temperature
Z_T	dimensionless phase lag for temperature gradient		

also only focused on solving Fourier heat conduction equation with AWE.

On the other hand, finite element method (FEM) has been extensively used to solve thermal problems because it is capable to account for complicated three-dimensional geometry. Besides that, the governing equations for Fourier and non-Fourier heat conduction are also parabolic and hyperbolic in nature, respectively, and they are difficult to be solved analytically. Using FEM, the transient heat conduction equation (partial differential equation) is reduced to a set of linear differential equations through the process of discretization. This set of differential equations can then be solved in time domain to obtain its transient solution.

Usually, this set of equations is solved using conventional iterative solvers such as Crank–Nicolson, Runge–Kutta and the famous Newmark algorithm. These conventional numerical solvers require the whole set of equations to be solved at each increment of time step, even though only the solution at a particular node is of interest. Solving this large set of equations is very time consuming, especially when the time step required is also very small in order to yield accurate solutions.

In contrast, AWE is actually approximating the original system with reduced order system and thus, it is a few orders faster than conventional iterative solvers in term of computational time. It is also independent of time step because it produces the transient solutions in a form of equation, rather than numerical solutions at every increment of time step. AWE is also capable of producing local solution because it can obtain the solution for each node independently and thus further reducing the amount of computational time. However, the drawback of AWE is that the moment matching process in AWE is inherently ill-conditioned and thus may produce unstable response even for stable system [10]. Higher order approximation will lead to a more accurate solution but not always guarantee a stable solution.

In this paper, FEM is coupled with AWE to efficiently solve the transient Fourier and non-Fourier heat conduction equations. FEM is used to reduce the Fourier (parabolic) and non-Fourier (hyperbolic) equations to a set of first and second order linear differential equations, respectively. AWE is then used to obtain the transient solutions instead of using conventional iterative solvers. The inherent instability of AWE is also addressed and two stability schemes are also introduced to yield accurate and yet stable solution even using higher order approximation.

2. Mathematical model for Fourier and non-Fourier heat conduction

Classical Fourier's law is based on diffusion model with assumption of infinite thermal wave propagation speed, which leads to simultaneous development of heat flux and temperature gradient. Classical Fourier's law also assumes that instantaneous local thermal equilibrium occurs between electrons and phonons. In other words, classical Fourier's law dictates that the thermal effect is felt instantaneously throughout the system if the surface of a material is heated. The governing equation for non-dimensionalized two-dimensional Fourier heat conduction is a parabolic equation as shown by Eq. (1).

$$\frac{\partial^2\theta}{\partial\delta^2}+\frac{\partial^2\theta}{\partial\varepsilon^2}=\frac{\partial\theta}{\partial\beta} \tag{1}$$

where θ is the dimensionless temperature and β is the dimensionless time. The dimensionless distance x and y are represented by δ and ε, respectively.

After discretizating Eq. (1) with Galerkin's weighted residual method, a set of first order linear differential equations is obtained as given by Eq. (2). The detailed formulations of Eq. (2) can be obtained from Logan [11].

$$C\dot{\theta}+K\theta=f \tag{2}$$

where C is known as the capacitive matrix, while K is the conductivity matrix. f represents the load vector, which can be time-dependent or time-independent.

Classical Fourier law is sufficient for most heat conduction phenomena, but it is inadequate to describe rapid heating response, such as VLSI interconnection heating. Thus, many non-Fourier heat conduction equations are proposed by many researchers to account for the finite thermal wave propagation speed and/or finite relaxation time to establish local thermal equilibrium between electrons and phonons. The non-Fourier model discussed in this paper is a two-phase lag model proposed by Tzou

[12]. Cheah et al. [13] has generalized it into a non-dimensionalized two-dimensional hyperbolic equation given by Eq. (3).

$$\frac{\partial^2\theta}{\partial\delta^2}+\frac{\partial^2\theta}{\partial\varepsilon^2}+Z_T\frac{\partial^3\theta}{\partial\beta\partial\delta^2}+Z_T\frac{\partial^3\theta}{\partial\beta\partial\varepsilon^2}=\frac{\partial\theta}{\partial\beta}+Z_q\frac{\partial^2\theta}{\partial\beta^2} \tag{3}$$

where

$$\theta=\frac{T-T_o}{T_w-T_o},\quad \beta=\frac{t}{l^2/\alpha},\quad \delta=\frac{x}{l},\quad \varepsilon=\frac{y}{h},$$
$$Z_T=\frac{\tau_T}{l^2/\alpha}\quad\text{and}\quad Z_q=\frac{\tau_q}{l^2/\alpha}$$

τ_T is the phase lag of spatial temperature gradient with respect to the local temperature (to account for finite relaxation time for electron–phonon equilibrium), while τ_q is the phase lag of heat flux with respect to the local temperature (to account for finite thermal wave speed). The dimensionless variables for these phase lags are represented by Z_T and Z_q, respectively. The length and width are given by l and h, respectively, whereas α is the thermal diffusivity.

Cheah et al. [13] has also discretized Eq. (3) using Galerkin's weighted residual method to form a set of second order linear differential equations, as shown by Eq. (4).

$$A\ddot{\theta}+C\dot{\theta}+K\theta=f \tag{4}$$

where

$$A_{\text{element}}=\frac{Z_q\bar{A}}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$$

$$C_{\text{element}}=\frac{\bar{A}}{12}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}+\frac{Z_T}{4\bar{A}}\left\{\begin{bmatrix}b_i^2&b_ib_j&b_ib_k\\b_jb_i&b_j^2&b_jb_k\\b_kb_i&b_kb_j&b_k^2\end{bmatrix}+\begin{bmatrix}c_i^2&c_ic_j&c_ic_k\\c_jc_i&c_j^2&c_jc_k\\c_kc_i&c_kc_j&c_k^2\end{bmatrix}\right\}$$

$$K_{\text{element}}=\frac{1}{4\bar{A}}\left\{\begin{bmatrix}b_i^2&b_ib_j&b_ib_k\\b_jb_i&b_j^2&b_jb_k\\b_kb_i&b_kb_j&b_k^2\end{bmatrix}+\begin{bmatrix}c_i^2&c_ic_j&c_ic_k\\c_jc_i&c_j^2&c_jc_k\\c_kc_i&c_kc_j&c_k^2\end{bmatrix}\right\}$$

3. The AWE algorithm

The concept of AWE is to approximate the original response of a system with a reduced order system. Fig. 1 shows the flow of AWE algorithm, which can be categorized into three major steps. The response of a system can be represented by a polynomial equation in s-domain, where the coefficients of this polynomial are known as the moments [2,3]. In moment generation, the moments are determined for zero state response (ZSR) and zero input response (ZSR). The concept of ZSR and ZIR is used by Ooi et al. [9] to account for the boundary and initial conditions, respectively. In ZSR, the initial conditions of the system are assumed to be zero, while the forcing functions are assumed to be zero in ZIR. In moment matching, the order of the system response is reduced using Padé approximation, and then further simplified to a set of partial fractions, where each partial fraction contains a pole and also a zero. Finally, each partial fraction is inversed Laplace back to time domain and summed up to provide the transient solution.

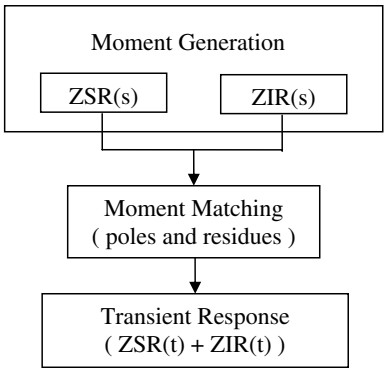

Fig. 1. Flow of AWE algorithm.

3.1. First order differential equation

As discussed in Section 2, parabolic heat conduction equation can be discretized into a set of first order linear differential equations using Galerkin's weighted residual method.

$$C\dot{T}+KT=F(t) \tag{5}$$

where $C\in R^{N\times N}$, $K\in R^{N\times N}$ and $F(t)\in R^N$.

Taking Laplace transform of Eq. (5),

$$C(sT(s)-T(0))+KT(s)=f \tag{6}$$

The system solution, $T(s)$ can be approximated by using polynomial equation in s-domain, as given by Eq. (7).

$$T(s)=\sum_{n=0}^{\infty}M_ns^n \tag{7}$$

The moments, M_n, are the coefficients of Taylor series expansion about $s=0$ (Maclaurin series) [2–4]. Moments are generated, respectively, for ZSR and ZIR by substituting Eq. (7) into Eq. (6) as following.

3.1.1. Zero state response (ZSR)

In ZSR, the initial condition is assumed to be zero, $T(0)=0$.

$$(Cs+K)(M_0+M_1s+M_2s^2+\cdots+M_ns^n)=f \tag{8}$$

By equating the same powers of s, the moments are generated from Eq. (9).

$$\begin{aligned}&KM_0=f\\&KM_n=-CM_{n-1}\quad\text{for } n=1,2,3,\ldots,(2q-1)\end{aligned} \tag{9}$$

where $q=$ order of Padé approximation.

3.1.2. Zero input response (ZIR)

In ZIR, the forcing function is assumed to be zero, $f = 0$.

$$(Cs + K)(M_0 + M_1 s + M_2 s^2 + \cdots + M_n s^n) = CT(0) \quad (10)$$

Again by equating the same powers of s, the moments are generated from Eq. (11).

$$\begin{aligned} KM_0 &= CT(0) \\ KM_n &= -CM_{n-1} \quad \text{for } n = 1, 2, 3, \ldots, (2q-1) \end{aligned} \quad (11)$$

3.2. Second order differential equation

For hyperbolic heat conduction equation as shown by Eq. (3), it can be reduced to a set of second order differential equations using Galerkin's weighted residual method [13,14].

$$A\ddot{T} + C\dot{T} + KT = F(t)$$

where $A \in R^{N\times N}$, $C \in R^{N\times N}$, $K \in R^{N\times N}$ and $F(t) \in R^N$.

Following the formulation above, the moments for ZSR and ZIR can be obtained from Eqs. (12) and (13), respectively.

3.2.1. Zero state response (ZSR)

$$\begin{aligned} KM_0 &= f \\ KM_1 &= -CM_0 \\ KM_n &= -(AM_{n-2} + CM_{n-1}) \quad \text{for } n = 2, 3, 4, \ldots\ldots, (2q-1) \end{aligned} \quad (12)$$

3.2.2. Zero input response (ZIR)

$$\begin{aligned} KM_0 &= CT(0) + A\dot{T}(0) \\ KM_1 &= AT(0) - CM_0 \\ KM_n &= -(AM_{n-2} + CM_{n-1}) \quad \text{for } n = 2, 3, 4, \ldots\ldots, (2q-1) \end{aligned} \quad (13)$$

3.3. Moment matching

In moment matching, only the moments for an arbitrary node of interest, i are used for computing its local transient response. This further reduces the computational time, in contrast to conventional iterative solver which requires computing the solutions of all the nodes at all time. The nodal moments of interest, $[m]$, are extracted from global moment matrix, $[M]$ as shown by Eq. (14).

$$[m_n]_i = [M_n]_i \quad \text{for } n = 0, 1, 2, \ldots, (2q-1) \quad (14)$$

The transient response at an arbitrary node of interest, i can be approximated by a lower order polynomial fraction using Padé approximation as shown in Eq. (15) and can be further simplified to partial fractions as given by Eq. (16).

$$\begin{aligned} T_i(s) &= m_0 + m_1 s + m_2 s^2 + \cdots + m_n s^n \\ &= \frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q} \end{aligned} \quad (15)$$

$$\frac{b_0 + b_1 s + \cdots + b_{q-1} s^{q-1}}{1 + a_1 s + \cdots + a_q s^q} = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_q}{s - p_q} \quad (16)$$

The poles and residues of the respective partial fractions can be obtained by solving Eqs. (17)–(19) [3,9].

$$\begin{bmatrix} m_0 & m_1 & m_2 & \cdot & m_{q-1} \\ m_1 & m_2 & m_3 & \cdot & m_q \\ m_2 & m_3 & m_4 & \cdot & m_{q+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{q-1} & \cdot & \cdot & \cdot & m_{2q-2} \end{bmatrix} \begin{Bmatrix} a_q \\ a_{q-1} \\ a_{q-2} \\ \cdot \\ a_1 \end{Bmatrix} = \begin{Bmatrix} -m_q \\ -m_{q+1} \\ -m_{q+2} \\ \cdot \\ -m_{2q-1} \end{Bmatrix} \quad (17)$$

$$\sum_{j=1}^{q} (a_j p^j) + 1 = 0 \quad (18)$$

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & \cdot & p_q^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & \cdot & p_q^{-2} \\ p_1^{-3} & p_2^{-3} & p_3^{-3} & \cdot & p_q^{-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_1^{-q} & p_2^{-q} & p_3^{-q} & \cdot & p_q^{-q} \end{bmatrix} \begin{Bmatrix} k_1 \\ k_2 \\ k_3 \\ \cdot \\ k_q \end{Bmatrix} = \begin{Bmatrix} -m_0 \\ -m_1 \\ -m_2 \\ \cdot \\ -m_{q-1} \end{Bmatrix} \quad (19)$$

3.4. Transient response

The transient response at an arbitrary node, i is given by the sum of ZSR and ZIR in time domain

$$T_i(t) = \mathrm{ZSR}(t) + \mathrm{ZIR}(t) \quad (20)$$

where

$$\mathrm{ZSR}(t) = \sum_{r=1}^{q} \frac{k_r}{p_r} (e^{p_r t} - 1) \quad (21)$$

$$\mathrm{ZIR}(t) = \sum_{r=1}^{q} k_r (e^{p_r t} - 1) \quad (22)$$

4. Unstable solutions and stability schemes

4.1. Inherent instability of AWE in Fourier heat conduction

The moment matching process in AWE is inherently ill-conditioned because Padé approximation is well-known for yielding unstable poles [10]. In the scope of finite element analysis, AWE can produce correct solutions for most of the nodes, but may yield incorrect solutions for some arbitrary nodes. A one-dimensional transient Fourier heat conduction example from Logan [11] is taken as case study. Using finite element method (FEM), it is discretized into 19 line elements (20 nodes) as shown in Fig. 2 and subjected to higher step temperature at the base as boundary condition. Solutions from AWE and also Crank–Nicolson (conventional iterative solver) agree well for all the nodes,

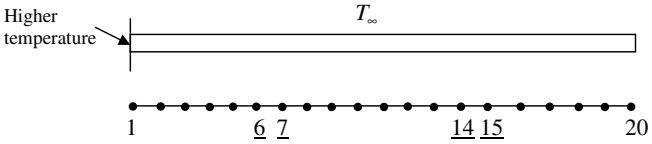

Fig. 2. Example from Logan [11] and its finite element representation.

except for nodes 6, 7, 14 and 15. This is because Padé approximation is known for producing unstable response even for stable system [15,16].

To further elaborate the weakness of Padé approximation, consider a stable system, $H(s)$, with two negative value poles and it is to be approximated by a one pole system, $G(s)$ using Padé approximation, as follows:

$$H(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} \quad \text{and} \quad G(s) = \frac{\bar{k}}{s - \bar{p}} \tag{23}$$

With some mathematical manipulation, it can be shown that $\bar{p}$ are related to the original poles and residues by Eq. (24) [17]. $\bar{p}$ has to be a negative value left-half plane pole in order to have a stable reduced order system, $G(s)$. However, as per Eq. (24), $\bar{p}$ can still be rendered to become a positive value right-half plane pole depending on the values of the residues (k_1 and k_2), even though both p_1 and p_2 are negative value left-half plane poles. The process of yielding undesirable positive poles is random, but it can be overcome by using the stability schemes as will be discussed in Sections 4.2 and 4.4 below.

$$\bar{p} = \frac{\frac{k_1}{p_1} + \frac{k_2}{p_2}}{\frac{k_1}{p_1^2} + \frac{k_2}{p_2^2}} \tag{24}$$

4.2. Stability schemes for Fourier heat conduction

AWE is based on approximating an original system with a reduced order system. So, higher order approximation is vital in ensuring that the exact response is successfully captured by the reduced order system. Besides yielding higher accuracy at the cost of slightly more computational time, higher order approximation is also prone to numerical instability. In most cases, stable and accurate solutions can be obtained by using lower order approximation. It was found that using 4th order Padé approximation is sufficient to provide accurate solutions for most Fourier heat conduction problems, and yet minimizing the number of unstable nodal solutions. However, this scheme may not always guarantee that the solutions are both stable and accurate for all the nodes, as in this case.

Fig. 3 shows that the AWE solution at node 14 starts to diverge from the Crank–Nicolson solution after 2.5 s. The divergence is due to unstable positive real poles. It is well-known in linear feedback control system that a system becomes unstable if any of the poles fall on the right-hand side of s-plane. Mathematically, poles with real positive

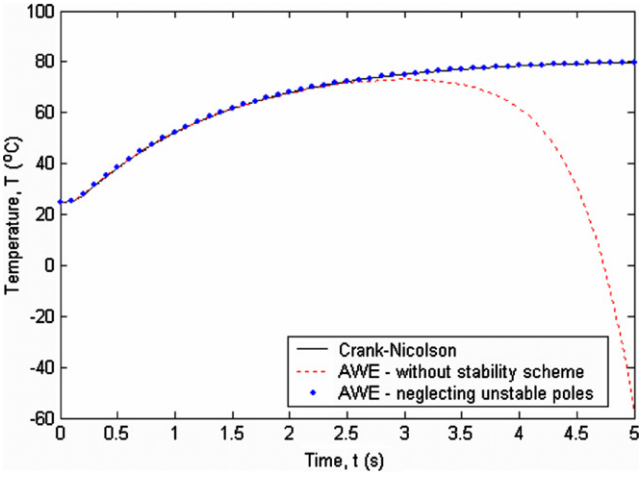

Fig. 3. Comparison between Crank–Nicolson and AWE (before and after applying stability scheme) solutions at node 14.

value will cause magnitude of ZSR or ZIR to become very large over time, and thus causing the solution to differ greatly from the correct response.

As given by Eqs. (21) and (22), ZSR and ZIR consist of the term $e^{p_r t}$. If any of the pole, p, has only real positive value, then the system response will be unable to converge to a steady-state solution. Instead, it will keep on increasing with time and consequently deviates from the correct transient response. Thus, it is reasonable to just simply neglect these unstable real positive poles [18]. This stability scheme proves to be valid as shown in Fig. 3, where the AWE solution agrees well with Crank–Nicolson solution even after 2.5 s.

4.3. Inherent instability of AWE in non-Fourier heat conduction

AWE is extended to model the transient non-Fourier heat conduction of two-dimensional slab subjected to instantaneous temperature rise on one edge [13], as shown in Fig. 4. The dimensionless parameters used are $\delta = \varepsilon = 1$(size) and $Z_q = 0.05$. Z_T is taken at three different values, which are 0.5 (over-diffusion), 0.05 (diffusion) and 0.0001(wave-like). Without implementing any stability scheme, some arbitrary nodes are expected to have positive real poles and the solutions at these nodes are going to be unstable and incorrect. As explained above, the generation of unstable positive real poles is due to the inherently ill-conditioned moment matching process in AWE.

In this case, higher order approximation is also more prone to yielding unstable response, whereas lower order approximation will reduce the number of nodes with unstable response. Similar to Fourier heat conduction problems, it was found that 4th order Padé approximation is best at yielding sufficiently accurate solutions with minimum number of nodes with unstable response for most non-Fourier heat conduction problems.

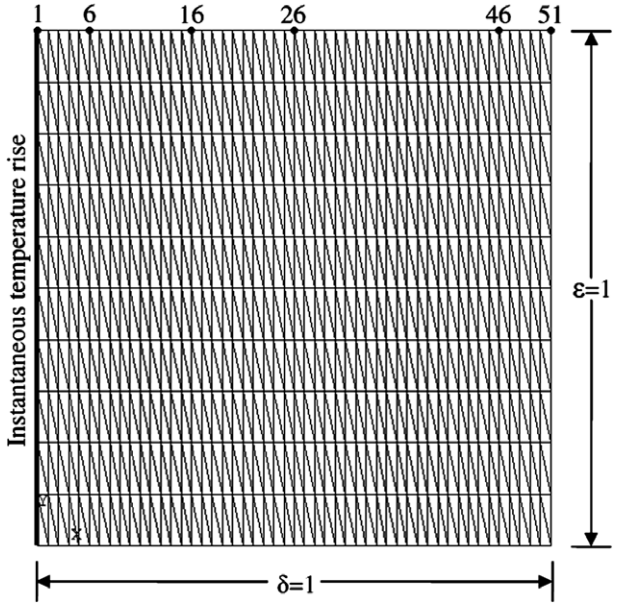

Fig. 4. Two-dimensional slab subjected to instantaneous temperature rise on left edge.

4.4. Stability schemes for non-Fourier heat conduction

The response of Fourier heat conduction has been successfully stabilized by neglecting the unstable real positive poles, as discussed in Section 4.2. Thus, this stability scheme is also implemented for two-dimensional non-Fourier heat conduction, in an attempt to stabilize all the temperature responses. The stabilized temperature responses for nodes along the top edge of the slab (from node 2 to node 51 as shown by Fig. 4) are plotted on Figs. 5–7 for three different values of Z_T, respectively. Figs. 5 and 6 do not show any unstable response along the slab, which indicates that this stability scheme has worked well for cases with $Z_T = 0.5$ and $Z_T = 0.05$.

However, AWE does not produce reasonable solutions for the case with $Z_T = 0.0001$ using this stability scheme.

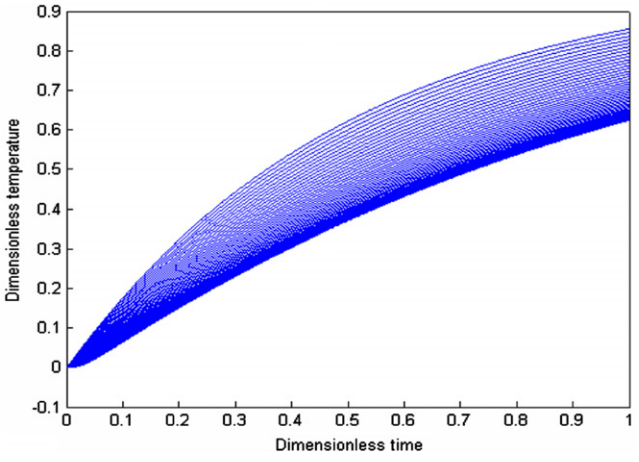

Fig. 5. Dimensionless temperature distribution along the top edge of slab at $Z_T = 0.5$.

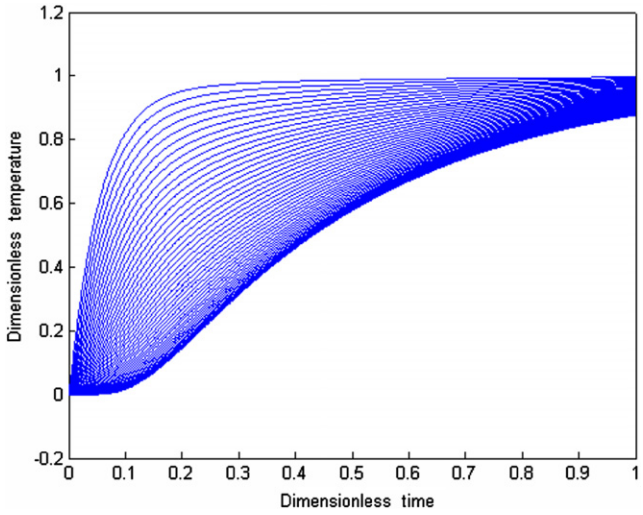

Fig. 6. Dimensionless temperature distribution along the top edge of slab at $Z_T = 0.05$.

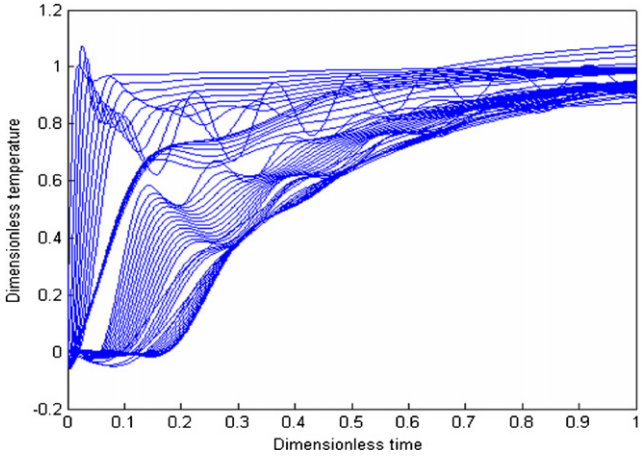

Fig. 7. Dimensionless temperature distribution along the top edge of slab at $Z_T = 0.0001$.

There is no smooth and continuous trend for the nodal responses along the top edge of slab, and the solutions at some nodes are also fluctuating incorrectly as shown in Fig. 7. The applied stability scheme does manage to suppress all unstable poles because there will be responses with very large magnitude if unstable poles exist. Yet, AWE solutions are still incorrect at some nodes because AWE fails to actually approximate the high frequency responses for nodes adjacent to the instantaneous temperature rise boundary condition, and thus causing the approximations for other nodes to deviate from the actual responses as well. In other words, AWE is incapable to fully represent the steep responses at some nodes since it is making use of only exponential terms (as shown by Eqs. (21) and (22)) to make approximation of the original response. Thus, this failure also renders the solutions at other nodes to be incorrect as well.

In addition, partial Padé approximation [15,16] is also introduced to stabilize the AWE solutions for the case

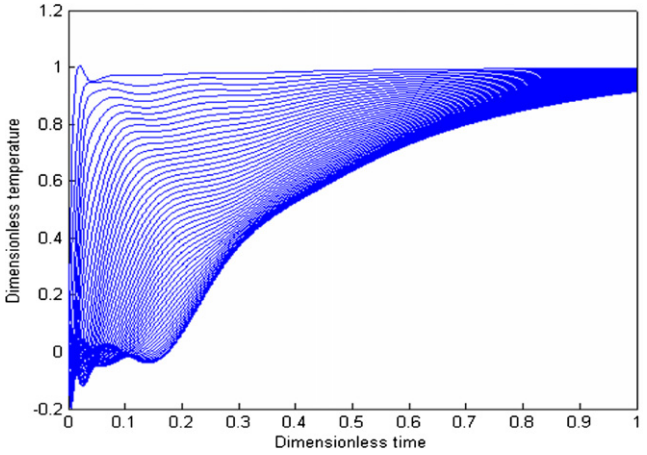

Fig. 8. Dimensionless temperature distribution along the top edge of slab at $Z_T = 0.0001$ after applying partial Padé approximation.

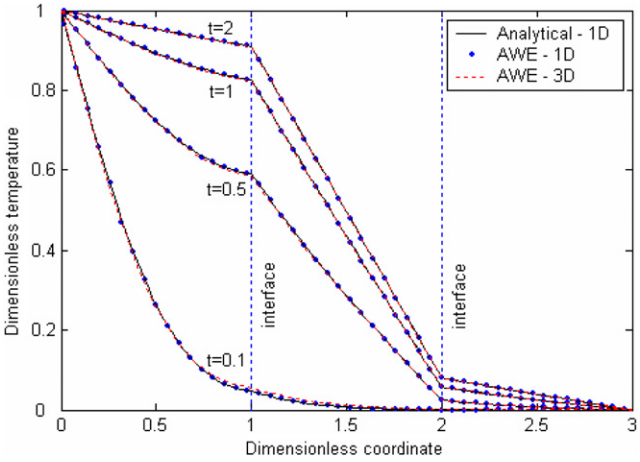

Fig. 9. Comparison between AWE and eigenfunction expansion method for dimensionless temperature across the layers at different dimensionless time.

$Z_T = 0.0001$. In this stability scheme, the set of poles at an arbitrarily chosen node is used to approximate the responses of other nodes. In other words, this means that only the set of poles at an arbitrarily chosen node is calculated and it is used throughout the calculations of ZSR and ZIR for all other nodes. Usually, the node selected to approximate other nodes has high frequency response, and definitely it has to be stable. Therefore, it will eliminate the need to monitor the stability at every node.

In this case, the node closest to the temperature load is chosen to approximate the temperature rise for other nodes along the top edge of slab, and only the poles at this chosen node are calculated. This node is chosen because it has the fastest temperature rise, or in other words, the response at this node is of high frequency. The temperature responses for nodes along the top edge of the slab are plotted on Fig. 8. There is no unreasonable solution as compared to the former stability scheme because the poles used are selected from a chosen node, which is stable and also representing the correct response. This eliminates the generation of incorrect poles at some nodes (when using former stability scheme), which may renders the solutions to be incorrect. One way to determine a suitable node is to check the responses by using former stability scheme before applying partial Padé approximation. The accuracy of this scheme will be discussed in Section 5.2.

5. Performance of AWE with stability schemes

5.1. Fourier heat conduction

Analytical solution is always more preferred because it provides an algebraic relationship equation that will generate solutions instantaneously, just by inputting the related parameter values. However, analytical method may not be viable for problems involving complex geometry and in most cases, it will also involve complicated mathematical derivations to arrive at the solution. One example is the eigenfunction expansion method, which has been used to solve the transient heat conduction for a one-dimensional three layers composite slab [19]. This method involves complicated mathematical derivations and also requires Newton iterative procedure to solve a non-linear equation for obtaining the eigenvalues. When more parameters (more layers) are involved, the eigenvalues become more difficult to obtain as the nonlinear equation becomes more complicated to be solved.

In comparison, finite element method (FEM) coupled with AWE as solver can provide accurate solution in a more efficient manner. FEM can be easily implemented to solve one-dimensional thermal problem, and also can be extended to solve two-dimensional or three-dimensional problems. AWE is also capable of providing the solution in equation form for each node. Besides that, moments in AWE are generated through direct solving of a system of linear equations. This will not be an issue even if the number of equations involved is large.

Fig. 9 compares the eigenfunction expansion solution with AWE solution for the one-dimensional three layers composite slab. It shows that AWE solutions are identical to the analytical solutions. In addition, this one-dimensional three layers composite slab is also extended into a three-dimensional problem with cross-sectional area of one and solved using AWE. Fig. 9 shows that there is slight difference between one-dimensional solution and three-dimensional solution (nodal temperatures taken across the centers of layers) because in-plane heat conduction effect is not considered in formulation for one-dimensional heat conduction.

5.2. Non-Fourier heat conduction

In Section 4.3, it is shown that both stability schemes are capable to suppress the generation of unstable poles for case $Z_T = 0.0001$. However, only partial Padé approximation can yield reasonable responses for all the nodes by using only one set of poles from a stable and reasonable

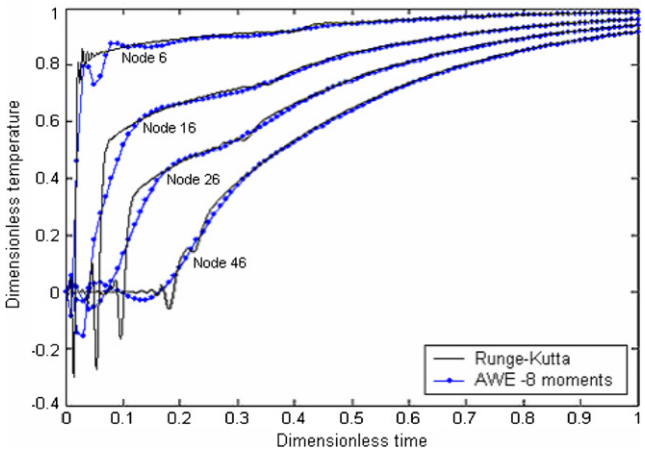

Fig. 10. Comparison between AWE (after applying partial Padé approximation) and Runge–Kutta for case $Z_T = 0.0001$.

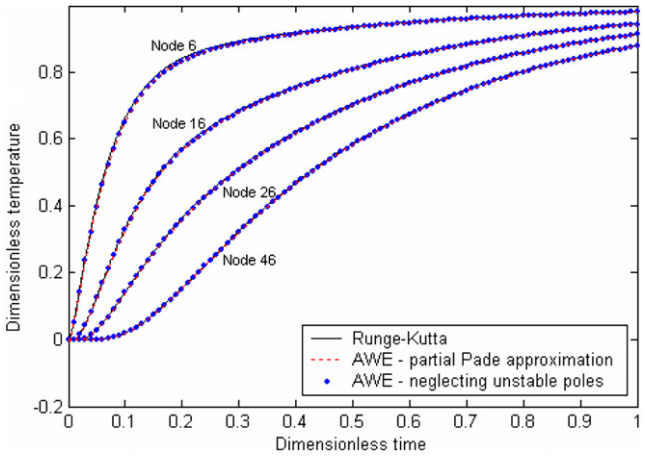

Fig. 11. Comparison between AWE (after applying stability schemes) and Runge–Kutta for case $Z_T = 0.05$.

chosen node. In order to check the accuracy, it is compared with Runge–Kutta solutions at node 6, 16, 26 and 46 along the slab. Positions of these nodes on the slab are shown in Fig. 4.

Runge–Kutta fourth order method has been used by Cheah et al. [13] to obtain his results. Fig. 10 shows that AWE solutions differ from the Runge–Kutta solutions at the initial time and only agree well approximately after 0.2 dimensionless time. This is expected because the concept of AWE is to extract the dominant poles from the full system [18] and these dominant poles are usually of low frequency. Thus, AWE responses are not able to match the high frequency temperature responses at the initial time for case $Z_T = 0.0001$. In other words, initial steep response cannot be approximated by AWE using exponential terms, as shown in Eq. (21).

In addition, Fig. 10 also shows that AWE is not able to approximate the delay accurately since the response only consists of real and complex conjugate exponentials. This is because, to represent a pure delay, AWE is employing decaying sinusoids to artificially force the response close to zero for some initial period. This will lead to spurious ringing effects in the AWE waveforms [18] as shown in Fig. 10.

For case $Z_T = 0.05$ and $Z_T = 0.5$, the AWE solutions can be successfully stabilized with both stability schemes, which are neglecting unstable real positive poles and partial Padé approximation. For accuracy check, AWE solutions after applying these stability schemes are compared with Runge–Kutta solutions. For case $Z_T = 0.05$, the comparison is shown on Fig. 11, where the results from using both stability schemes agree well with Runge–Kutta solutions, respectively. AWE shows similar accuracy for case $Z_T = 0.5$ as well, but the results are not depicted in this paper.

In term of computational time, AWE only needs 1.2 s for case $Z_T = 0.0001$ while Runge–Kutta requires 31 min even though the same total number of nodes is used. For case $Z_T = 0.05$, AWE can produce the result in less than 1.5 s using both stability schemes while Runge–Kutta requires 1.8 hours. In this case, Runge–Kutta requires approximately 3.5 times more than for case $Z_T = 0.0001$ because smaller time step is required to attain convergence. However, AWE can still produce the solutions at approximately the same amount of computational time for both cases. This is because AWE is independent of time step size used. Thus, AWE is much more efficient than Runge–Kutta in terms of speed but still it can produce accurate results equivalent to Runge–Kutta.

6. Conclusion

AWE has proven itself to be a powerful solver for fast transient thermal characterization of Fourier and non-Fourier heat conduction. When AWE is coupled with FEM, it is capable of solving one-dimensional, two-dimensional and even three-dimensional heat conduction problems. AWE is also at least three orders faster in terms of computational time as compared to conventional iterative solvers, but still providing equivalent accuracy. The drawback of AWE is its inherent instability of Padé approximation, which may yield unstable solutions even for stable system. However, this instability can be overcome by applying any of the two stability schemes that has been introduced in this paper. Besides that, AWE does have the limitation of approximating very high frequency response for non-Fourier heat conduction case with $Z_T = 0.0001$. This limitation is indicated by incorrect responses even after neglecting the unstable positive poles, and thus AWE is not recommended for use. Even though applying partial Padé approximation may help to yield reasonable responses, but the initial high frequency response still cannot be approximated accurately.

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